

A Stochastic Game Model for the Optimal Timing of Regional Climate Policies

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Talk overview

In this presentation, we **revisit** first:

- Bahn, O., Haurie, A., Malhamé, R. (2008). "A stochastic control model for optimal timing of climate policies", *Automatica*, Vol. 44, pp. 1545-1558.

in a **cost-benefit framework**.

We **extend** then:

- Bahn, O., Haurie, A. (2008). "A class of games with coupled constraints to model international GHG emission agreements", *International Game Theory Review*, Vol. 10, pp. 337-362.

to a **stochastic game approach**.

Outline

- 1 Stochastic Control Approach
 - Economic modeling
 - Climate change dynamics
 - Stochastic control formulation
 - Dynamic programming
 - Numerical results
- 2 Stochastic Game Approach
 - Multi-region modeling
 - Game in normal form
 - Dynamic programming
 - Numerical results
- 3 Conclusion

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Two economies

We distinguish between two types of economy:

- a "carbon economy", where a high level of carbon emissions is necessary to obtain output
- a "carbon-free economy", where a much lower level of emissions is necessary to produce the economic good.

We use two distinct types of capital for the two economies.

The carbon-free capital cannot initially contribute to the production of economic output: the associated technology must first become available through an uncertain "breakthrough".

R&D investments in the carbon-free technology will increase the probability that such a breakthrough takes place.

Variables-1

- $C(t) \geq 0$: total consumption at time t , in trillions (10^{12}) of dollars;
- $E(t) \geq 0$: global yearly emissions of GHG at time t , in Gt (10^9 tons) carbon equivalent;
- $E_1(t) \geq 0$: part of the emissions attributed to the carbon economy;
- $E_2(t) \geq 0$: part of the emissions attributed to the carbon-free economy;
- $ELF(t) \geq 0$: economic loss factor at time t due to climate changes, in %;
- $I_i(t) \geq 0$: investment at time t in capital $i = 1, 2$, in trillions of dollars;
- $K_1(t) \geq 0$: physical stock of productive capital in the carbon economy at time t , in trillions of dollars;
- $K_2(t) \geq 0$: physical stock of (R&D or productive) capital in the carbon-free economy at time t , in trillions of dollars;

Variables-2

$L(t) \geq 0$: labor (world population), in millions (10^6) persons;

$L_1(t) \geq 0$: part of the labor force allocated to the carbon economy;

$L_2(t) \geq 0$: part of the labor force allocated to the carbon-free economy;

t : running time, (unit = 10 years, in our discrete time formulation);

$\xi(t) \in \{0, 1\}$: indicator variable for the carbon-free economy, where $\xi(t) = 0$ indicates that $K_2(t)$ is an R&D capital;

W : total discounted welfare;

$Y(t)$: economic output at time t , in trillions of dollars.

Economic dynamics-1

$$W = \int_0^{\infty} e^{-\rho t} L(t) \log[C(t)/L(t)] dt$$

$$L(t) = L_1(t) + L_2(t)$$

$$E(t) = E_1(t) + E_2(t)$$

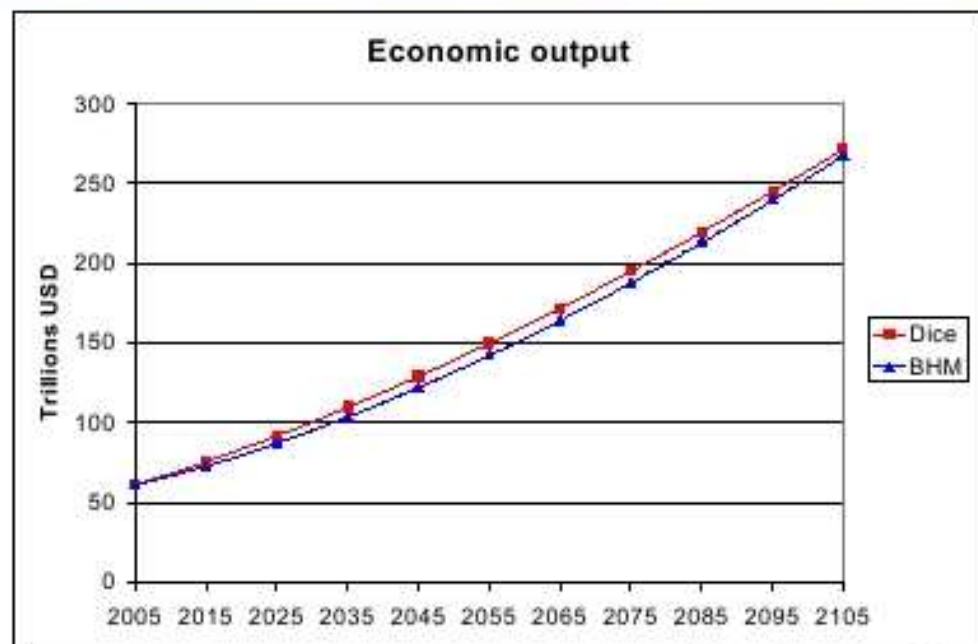
$$Y(t) = A_1(t)K_1(t)^{\alpha_1}(\phi_1(t)E_1(t))^{\theta_1(t)}L_1(t)^{1-\alpha_1-\theta_1(t)} \\ + \xi(t)A_2(t)K_2(t)^{\alpha_2}(\phi_2(t)E_2(t))^{\theta_2(t)}L_2(t)^{1-\alpha_2-\theta_2(t)}$$

$$\text{ELF}(t)Y(t) = C(t) + I_1(t) + I_2(t) \\ + p_{E1}(t)\phi_1(t)E_1(t) + p_{E2}(t)\phi_2(t)E_2(t)$$

Economic dynamics-2

$$\begin{aligned}
 \dot{L}(t) &= g_L(t)L(t) \\
 \dot{g}_L(t) &= -\delta_{g_L} g_L(t) \\
 \dot{A}_i(t) &= g_{A_i}(t)A_i(t); \quad i = 1, 2 \\
 \dot{g}_{A_i}(t) &= -\delta_{g_{A_i}} g_{A_i}(t); \quad i = 1, 2 \\
 \dot{K}_1(t) &= I_1(t) - \delta_{K_1} K_1(t) \\
 \dot{K}_2(t) &= I_2(t) - \delta_{K_2} K_2(t) \\
 \dot{\phi}_i(t) &= g_{\phi_i}(t)\phi_i(t); \quad i = 1, 2 \\
 \dot{g}_{\phi_i}(t) &= -\delta_{g_{\phi_i}} g_{\phi_i}(t); \quad i = 1, 2 \\
 \dot{\theta}_i(t) &= g_{\theta_i}(t)\theta_i(t); \quad i = 1, 2 \\
 \dot{g}_{\theta_i}(t) &= -\delta_{g_{\theta_i}} g_{\theta_i}(t); \quad i = 1, 2 \\
 \dot{p}_{E_i}(t) &= e^{\rho t} p_{E_i}(t); \quad i = 1, 2.
 \end{aligned}$$

Calibration-1: BAU solution compared to DICE-2007



Calibration-2: Carbon vs carbon-free economy

Carbon economy

$\phi_1(0)$: initial value for energy efficiency; $\phi_1(0)=1.0$;

$g_{\phi_1}(0)$: initial value for growth rate of energy efficiency; $g_{\phi_1}(0) = 0.15$ per decade;

$\delta_{g_{\phi_1}}$: rate of decrease for growth rate of energy efficiency;
 $\delta_{g_{\phi_1}} = -0.2$;

p_{E_1} : price of energy input; $p_{E_1}(0) = 0.35$;

Carbon-free economy

$\phi_2(0)$: initial value for energy efficiency; $\phi_2(0)=5.0$;

$g_{\phi_2}(0)$: initial value for growth rate of energy efficiency; $g_{\phi_2}(0) = 0.15$ per decade;

$\delta_{g_{\phi_2}}$: rate of decrease for growth rate of energy efficiency;
 $\delta_{g_{\phi_2}} = -0.2$;

p_{E_2} : price of energy input; $p_{E_2}(0) = 0.6$.

Technological breakthrough dynamics-1

Variable: To describe the access to the cleaner productive system a binary variable $\xi \in \{0, 1\}$ is introduced which indicates if the clean technology is available ($\xi = 1$) or not yet ($\xi = 0$).

Breakthrough dynamics: The initial value $\xi(0) = 0$ indicates that there is no access to the clean capital at the initial time. The switch to the value 1 occurs at a random time which is controlled through the global accumulation of R&D capital K_2 .

Technological breakthrough dynamics-2

Elementary probability of a breakthrough:

$$P[\xi(t + dt) = 1 | \xi(t) = 0, K_2(t)] = q_b(t, K_2(t)) dt + o(dt).$$

One uses an affine form for the jump rate function:

$$q_b(t, K_2(t)) = \omega_b + v_b \sqrt{K_2(t)}.$$

In the above equation, parameters are defined as follows:

ω_b : initial probability rate of discovery; $\omega_b = 0.05$;

v_b : slope w.r.t $K_2(t)$ of the probability rate of discovery; $v_b = 0.0019$.

Carbon dynamics

Let $M(t)$ define atmospheric concentration of GHG at time t , in GtC equivalent. The accumulation of GHG in the atmosphere is described by the following equation:

$$\dot{M}(t) = \beta E(t) - \delta_M (M(t) - M_p),$$

where $M(0) = 808.8$ GtC and where parameters are defined as follows:

- β : marginal atmospheric retention rate; $\beta = 0.64$;
- δ_M : natural atmospheric elimination rate; $\delta_M = 0.036$;
- M_p : preindustrial level of atmospheric concentration;
 $M_p = 588.2$ GtC.

Climate change damages

To take damages related to climate change into account, we compute an economic loss factor (applied on economic output):

$$\text{ELF}(t) = 1 - \left(\frac{M(t) - M_d}{cat_M - M_d} \right)^2,$$

where

cat_M : “catastrophic” concentration level; $cat_M = 3.5$ times the pre-industrial concentration level;

M_d : level at which damages start to occur;
 $M_d = 735.3 \text{ GtC (350 ppm)}$.

Stochastic control model-1

- State variables are denoted $s = (\mathbf{K}, M, \xi)$ with $(\mathbf{K}, M) = x$.
- At initial time s^0 is such that $K_2^0 \simeq 0$ and $\xi^0 = 0$ (clean technology not yet available).
- Control variables are $u = (\mathbf{E}, \mathbf{I}, \mathbf{L})$ where $\mathbf{E}(t) = (E_i(t))_{i=1,2}$, $\mathbf{I}(t) = (I_i(t))_{i=1,2}$, $\mathbf{L}(t) = (L_i(t))_{i=1,2}$.
- Dynamics of the state variable x can be described through a control system with jump Markov disturbances:

$$\dot{x}(t) = f^{\xi(t)}(t, x(t), u(t))$$

Stochastic control model-2

- Given x and u , the instantaneous utility of consumption is determined. One can thus introduce the reward function:

$$\mathcal{L}^\xi(t, x, u) = L \log \left(\frac{1}{L} \left[F^\xi(t, \mathbf{E}, \mathbf{K}, \mathbf{L}) - \sum_{i=1}^2 (l_i + p_{E_i} \phi_i E_i) \right] \right),$$

where $F^\xi(t, \mathbf{E}, \mathbf{K}, \mathbf{L}) = \text{ELF } Y$.

- The controls are subject to the constraints: $E_i(t) \geq 0$, $l_i(t) \geq 0$, $L_i(t) \geq 0$, $i = 1, 2$ and $L_1(t) + L_2(t) \leq L(t)$. This is summarized in general notations by: $u(t) \in U(t)$.

Piecewise deterministic control model

- Consider the sequence of times τ^0, τ^1 where $\tau^0 = 0$ is the initial time and τ^1 is the random jump time of the $\xi(\cdot)$ process. Denote s^0, s^1 the state observed at times τ^0, τ^1 .
- A **piecewise deterministic** policy γ is a mapping that associates with τ^0 and an observed state s^0 a control:

$$u_{\tau^0, s^0}(\cdot) : (\tau^0, \infty[\rightarrow U(\cdot)$$

to be used until the jump occurs. At that point the policy must be updated to $u_{\tau^1, s^1}(\cdot) : (\tau^1, \infty[\rightarrow U(\cdot)$.

DP-1: After the jump

The jump occurs at time τ^1 . From that time onwards, the carbon-free technology is available.

At time τ^1 , for a given state $s^1 = (x^1, \xi^1) = (K^1, M^1, 1)$, let us define the value function $V^1(\tau^1, x^1)$ as the solution to the optimization problem:

$$V^1(\tau^1, x^1) = \max_{u(\cdot)} e^{\rho\tau^1} \int_{\tau^1}^{\infty} e^{-\rho t} \mathcal{L}^1(t, x(t), u(t)) dt \quad (1)$$

subject to the state equations:

$$\dot{x}(t) = f^1(t, x(t), u(t)); u(t) \in U(t); t \geq \tau^1; x(\tau^1) = x^1. \quad (2)$$

DP-2: Analytical approximations of V^1

- For a given grid of initial values (state space and last jump time), solve the deterministic control problem which corresponds to Eqs. (1)-(2).

This problem is formulated using GAMS and the nonlinear optimization problem is solved using CONOPT3.

- Then, based on a sufficient set of grid point evaluations, use a log-linear least squares algorithm to fit a product form:

$$a(\tau)K_1(t)^{b(\tau)}K_2(t)^{c(\tau)}M(t)^{d(\tau)}$$

DP-3: At the initial time-1

At initial time $\xi = 0$, that is one does not have access to the clean technology. The stochastic control problem to solve can be described as follows:

$$\begin{aligned}
 V^0(x^0) &= \max_{u(\cdot)} E_{K_2(\cdot)} \left[\int_0^{\tau^1} e^{-\rho t} \mathcal{L}^0(t, x(t), u(t)) dt \right. \\
 &\quad \left. + e^{-\rho \tau^1} V^1(\tau^1, x^1(\tau^1)) \right] \\
 &\quad \text{s.t.} \\
 \dot{x}(t) &= f^0(t, x(t), u(t)); \quad u(t) \in U(t); \quad t \geq 0; \quad x(0) = x^0.
 \end{aligned}$$

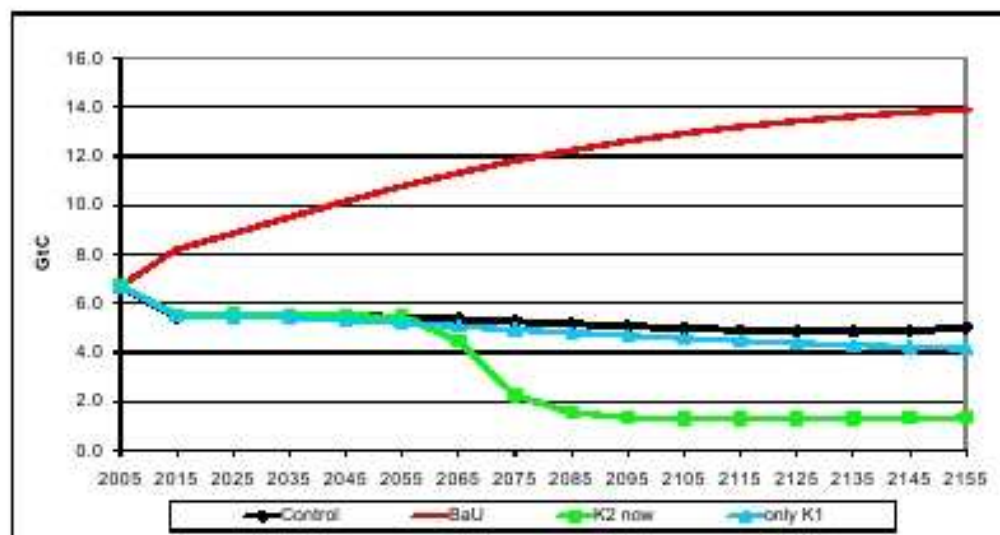
Here the jump time τ^1 is stochastic. The associated jump rate is $q_b(t, K_2(t))$. One has denoted $s^1(\tau^1)$ the random state reached after the jump time.

DP-4: At the initial time-2

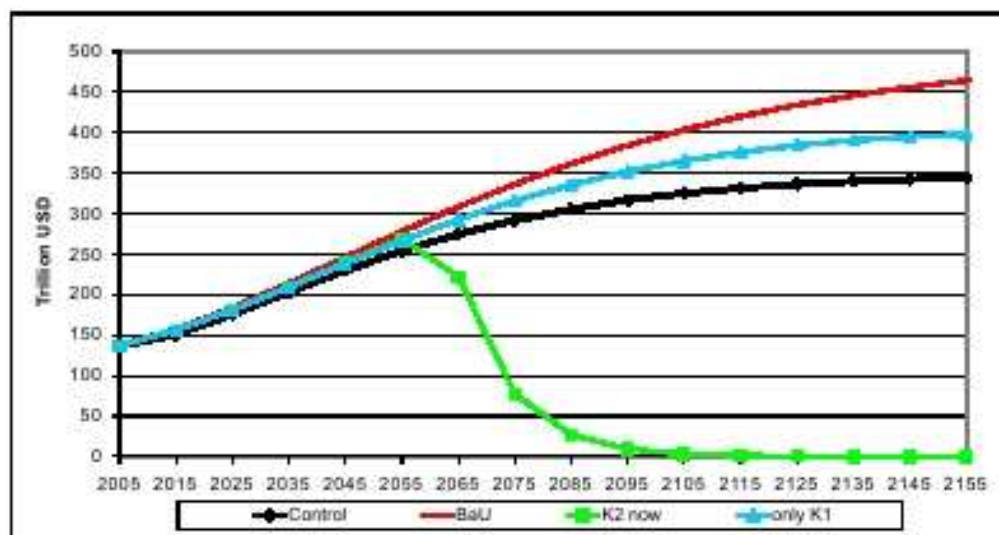
Using standard probability reasoning for this type of problem one obtains the equivalent **infinite horizon deterministic control problem**:

$$\begin{aligned}
 V^0(x^0) &= \max_{u(\cdot)} \int_0^\infty e^{-(\rho t + \int_0^t (q_b(s, K_2(s))) ds)} \left[\mathcal{L}^0(t, x(t), u(t)) \right. \\
 &\quad \left. + q_b(t, K_2(t)) V^1(t, x(t)) \right] dt \\
 &\quad s.t. \\
 \dot{x}(t) &= f^0(t, x(t), u(t)); \quad u(t) \in U(t); \quad t \geq 0; \quad x(0) = x^0.
 \end{aligned}$$

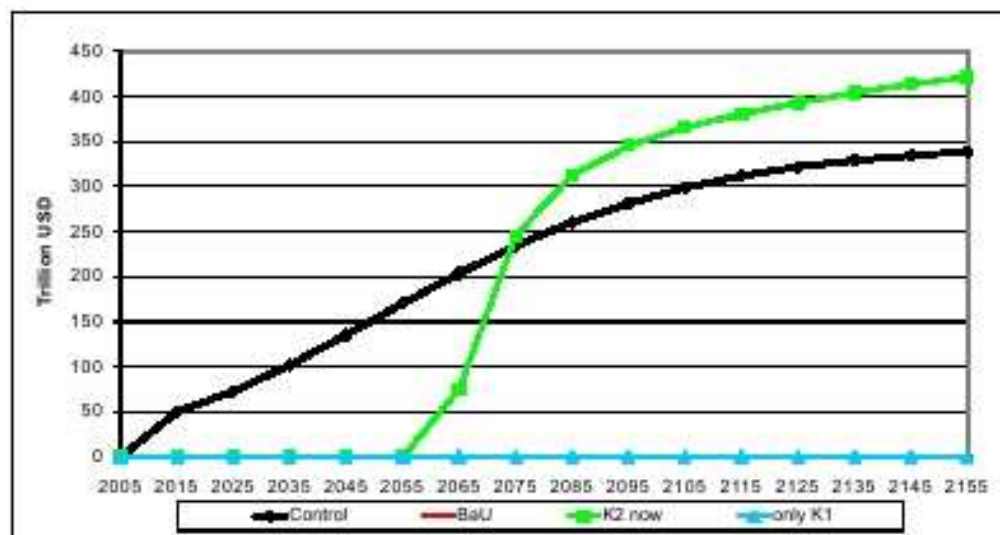
GHG emission trajectories



Dirty capital (K_1) trajectories



Clean capital (K_2) paths



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The following variables enter in the description of the economic model, where j is the index of each of the 2 regions and t corresponds to the model running time:

- $C(j, t)$: total consumption in region j at time t , in trillions (10^{12}) of dollars;
- $ELF(t)$: economic loss factor due to climate changes at time t , in %;
- $E_1(j, t)$: yearly emissions of GHG (in Gt– 10^9 tons–carbon equivalent) in the carbon economy of region j at time t ;
- $E_2(j, t)$: yearly emissions of GHG in the clean economy of region j at time t , in GtC;
- $I_i(j, t)$: investment in capital $i = 1, 2$ in region j at time t , in trillions of dollars;
- $K_1(j, t)$: physical stock of productive capital in the carbon economy of region j at time t , in trillions of dollars;
- $K_2(j, t)$: physical stock of productive capital in the clean economy of region j at time t , in trillions of dollars;
- $L_1(j, t)$: part of the (exogenously defined) labor force $L(j, t)$ of region j allocated at time t to the carbon economy, in millions (10^6) persons;
- $L_2(j, t)$: part of the labor force of region j allocated at time t to the clean economy, in millions persons;
- $M(t)$: atmospheric concentration of GHG at time t , in GtC equivalent;
- $WRG(j)$: discounted welfare of region j ;
- W : total discounted welfare;
- $Y_1(j, t)$: economic output in the carbon economy of region j at time t , in trillions of dollars;
- $Y_2(j, t)$: economic output in the clean economy of region j at time t , in trillions of dollars.

Economic dynamics

$$WRG(j) = \int_0^{\infty} e^{-\rho(j)t} L(j, t) \log[C(j, t)/L(j, t)] dt$$

$$L(j, t) = L_1(j, t) + L_2(j, t)$$

$$E(j, t) = E_1(j, t) + E_2(j, t)$$

$$Y(j, t) = A_1(j, t) K_1(j, t)^{\alpha_1(j)} (\phi_1(j, t) E_1(j, t))^{\theta_1(j, t)} L_1(j, t)^{1-\alpha_1(j)-\theta_1(j, t)} \\ + \xi(t) A_2(j, t) K_2(j, t)^{\alpha_2(j)} (\phi_2(j, t) E_2(j, t))^{\theta_2(j, t)} L_2(j, t)^{1-\alpha_2(j)-\theta_2(j, t)}$$

$$ELF(t) Y(j, t) = C(j, t) + h_1(j, t) + h_2(j, t) \\ + p_{E1}(j, t) \phi_1(j, t) E_1(j, t) + p_{E2}(j, t) \phi_2(j, t) E_2(j, t)$$

Technological breakthrough

Access to a cleaner economy at time t is represented by $\xi(t) \in \{0, 1\}$.
 There is no access to the clean technology at the initial time ($\xi(0) = 0$).
 Technological breakthrough occurs at a random time which is influenced by the global accumulation of R&D capital $\bar{K}_2$ where:

$$\bar{K}_2(t) = \sum_{j=1}^2 K_2(j, t).$$

The jump rate function is defined as:

$$q_b(t, \bar{K}_2(t)) = \omega_b + v_b \sqrt{\bar{K}_2(t)}.$$

Note that the research effort of both regions add up and that, after discovery, there is full spillover of the technological progress.

Climate change dynamics

Accumulation M of GHG in the atmosphere evolves according to:

$$\dot{M}(t) = \beta \sum_{j=1}^2 E(j, t) - \delta_M (M(t) - M_p),$$

As before, we assume that the economic loss factor ELF is defined by:

$$\text{ELF}(t) = 1 - \left(\frac{M(t) - M_d}{cat_M - M_d} \right)^2,$$

where cat_M corresponds to a "catastrophic" concentration level and M_d the concentration level at which damages start to occur, levels that are the same for the two regions.

Stochastic control model-1

- State variables are denoted $s(\cdot) = ((\mathbf{K}(j, \cdot) : j = 1, 2), M(\cdot), \xi(\cdot))$ with $(\mathbf{K}(\cdot), M(\cdot)) = x(\cdot)$.
- At initial time s^0 is such that $K_2^0 \simeq 0$ and $\xi^0 = 0$ (clean technology not yet available).
- Control variables are $u = (\mathbf{E}, \mathbf{I}, \mathbf{L})$ where $\mathbf{E}(t) = (E_i(j, t))_{i,j=1,2}$, $\mathbf{I}(t) = (I_i(j, t))_{i,j=1,2}$, $\mathbf{L}(j, t) = (L_i(j, t))_{i,j=1,2}$, for $j = 1, 2$.
- Dynamics of the state variable x can be described through a control system with jump Markov disturbances:

$$\dot{x}(t) = f^{\xi(t)}(t, x(t), u(t))$$

Stochastic control-2

- The reward function:

$$\mathcal{L}^\xi(j, t, x, u) = L(j, t) \log \left(\frac{1}{L(j, t)} \left[\text{ELF}(t) F^{\xi(t)}(j, t, \mathbf{E}(j, t), \mathbf{K}(j, t), \mathbf{L}(j, t)) - \sum_{i=1}^2 (l_i(j, t) + p_{E_i} \phi_i E_i(j, t)) \right] \right),$$

where $F^{\xi(t)}(j, t, \mathbf{E}(j, t), \mathbf{K}(j, t), \mathbf{L}(j, t)) = Y(j, t)$.

- The controls are subject to the constraints: $E_i(j, t) \geq 0$, $l_i(j, t) \geq 0$, $L_i(j, t) \geq 0$, $i = 1, 2$ and $L_1(j, t) + L_2(j, t) \leq L(t)$. This is summarized in general notations by: $u(t) \in U(t)$.

Piecewise open-loop strategies

- Consider the times τ^0, τ^1 where $\tau^0 = 0$ is the initial time and τ^1 the random jump time of the $\xi(\cdot)$ process. Denote s^0, s^1 the state observed at times τ^0, τ^1 .
- A piecewise open-loop strategy γ_j for Player j is a mapping that associates with τ^0 and an observed state s^0 a control $u_{j,\tau^0,s^0}(\cdot)$: $(\tau^0, \infty[\rightarrow U(j, \cdot))$ to be used by Player j until the jump occurs. At that point the policy must be updated.
- Associated with a policy vector $\gamma = \{\gamma_j : j = 1, 2\}$ and an initial state s^0 one obtains an expected reward (welfare gain) for Player j defined by:

$$J(j, \gamma, s^0) = E_\gamma \left[\int_0^\infty e^{-\rho(j)t} \mathcal{L}^{\xi(t)}(j, t, x(t), u(t)) dt \right],$$

where the expectation E_γ is taken with respect to the probability measure induced by the policy γ .

Nash equilibrium

Definition

A strategy γ^* is a Nash equilibrium if the following inequality holds:

$$J(j, \gamma^*, s^0) \geq J(j, (\gamma^{*-j}, \gamma_j), s^0) \quad (3)$$

for all initial state s^0 and Player j , where (γ^{*-j}, γ_j) stands for the policy vector where Player $i \neq j$ uses strategy γ_i^* whereas Player j uses an admissible strategy γ_j .

After the jump

Assume that the jump occurs at time τ^1 . From time τ^1 onwards, the carbon-free technology is available.

Let us define the value function $V^1(j, \tau^1, x^1)$ as the payoff to Player j in the equilibrium solution to an open-loop differential game:

$$V^1(j, \tau^1, x^1) = \max_{u(j, \cdot)} \left[e^{\rho(j)\tau^1} \int_{\tau^1}^{\infty} e^{-\rho(j)t} \mathcal{L}^1(j, t, x(t), [u(j, t), u^*(i, t)]) dt : \right. \\ \left. \text{given } u^*(i, \cdot); i \neq j \right]$$

subject to the state equations:

$$\dot{x}(t) = f^1(t, x(t), [u(j, t), u^*(i, t)]); \quad [u(j, t), u^*(i, t)] \in U(t); \quad t \geq \tau^1; \quad x(\tau^1) = x^1$$

where $u^*(i, \cdot)$ is the equilibrium control of the other player.

Fixed point of a response function

Denote: $\psi(j, \tau^1, s^1, u(\cdot)) = e^{\rho(j)\tau^1} \int_{\tau^1}^{\infty} e^{-\rho(j)t} \mathcal{L}^1(j, t, x(t), u(j, t)) dt$

Let us take any weighting $r = (r_j > 0 : j = 1, 2)$ and define the following function:

$$\theta(\tau^1, s^1, u(\cdot), v(\cdot); r) = \sum_{j=1}^2 r_j \psi(j, \tau^1, s^1, (u(\iota, \cdot), v(j, \cdot)))$$

where, as before, $\iota = 2$ if $j = 1$ and $\iota = 1$ if $j = 2$ and where $v(\cdot) \in U(\cdot)$.

Now define the generalized reply mapping as the point to set map:

$$\Theta(u(\cdot)) = \left\{ \tilde{v}(\cdot) : \theta(\tau^1, s^1, u(\cdot), \tilde{v}(\cdot); r) = \max_{v(\cdot) \in U(\cdot)} \theta(\tau^1, s^1, u(\cdot), v(\cdot); r) \right\}.$$

A fixed point $u^*(\cdot) \in \Theta(u^*(\cdot))$ is a Nash equilibrium for this differential game.

Cobweb approach

Let $G \subset \mathbb{R}^5$ be a grid for initial value $s^1 = (K_1^1(j), K_2^1(j), M^1)$ and let $\mathcal{T}$ be a set of initial times.

Step-0 For each $\tau^1 \in \mathcal{T}$ and each $s^1 \in G$

Step-1 find $u^*(\cdot) = \operatorname{argmax}_{u(\cdot)} \left[\sum_{j=1}^2 r_j \psi(j, \tau^1, s^1, u(\cdot)) \right]$

Step-2 take $u(\cdot) = u^*(\cdot)$; find

$$v^*(\cdot) = \operatorname{argmax}_{v(\cdot)} \left[\sum_{j=1}^2 r_j \psi(j, \tau^1, s^1, \{u(\cdot), v(j, \cdot)\}) \right]$$

Step-3 take $u(\cdot) = v^*(\cdot)$; find new

$$v^*(\cdot) = \operatorname{argmax}_{v(\cdot)} \left[\sum_{j=1}^2 r_j \psi(j, \tau^1, s^1, \{u(\cdot), v(j, \cdot)\}) \right]$$

Step-4 repeat step-3 until $\max_{v(\cdot)} \left[\sum_{j=1}^2 r_j \psi(j, \tau^1, s^1, \{u(\cdot), v(j, \cdot)\}) \right]$
 does not change significantly.

Step-5 record for each $j = 1, 2$ the value $V^1(j, \tau^1, x^1) = \psi(j, s, u^*(\cdot))$ where $s = (x^1, 1)$,
 $x^1 \in G$.

Step-6 Adjust by LSE an analytical form for the function $V^1(j, \tau^1, x^1)$, $x \in \mathbb{R}^5$.

At the initial time

At initial time the discrete state is $\xi(0) = 0$, that is one does not have access to the clean technology. The stochastic control problem describing the optimal response of player j to the equilibrium control of the other player i can be described as follows:

$$\begin{aligned}
 V^0(j, x^0) &= \max_{u(j, \cdot)} E_{K_2(\cdot)} \left[\int_0^{\tau^1} e^{-\rho(j)t} \mathcal{L}^0(j, t, x(t), [u(j, t), u^*(i, t)]) dt \right. \\
 &\quad \left. + e^{-\rho(j)\tau^1} V^1(j, \tau^1, x^1(\tau^1)) : \text{ given } u^*(i, \cdot); i \neq j \right] \\
 \text{s.t.} \\
 \dot{x}(t) &= f^0(t, x(t), [u(j, t), u^*(i, t)]); \quad [u(j, t), u^*(i, t)] \in U(t); \\
 t &\geq 0; \quad x(0) = x^0.
 \end{aligned}$$

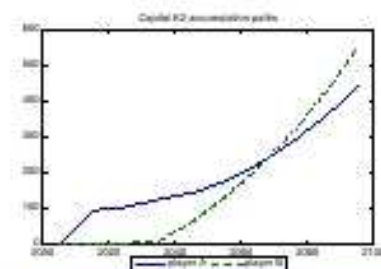
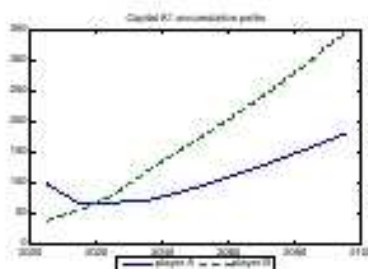
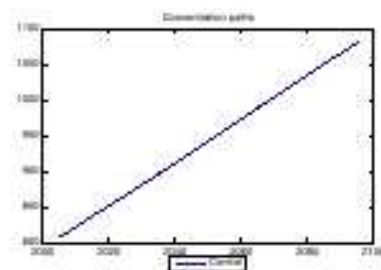
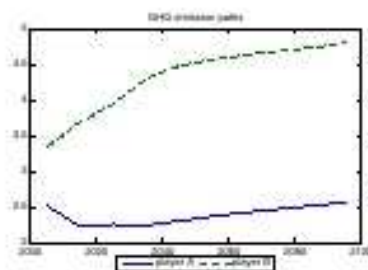
Deterministic equivalent problem

The associated infinite horizon differential game problem will be:

$$\begin{aligned}
 V^0(x^0) &= \max_{u(j, \cdot)} \int_0^\infty e^{-(\rho(j)t + \int_0^t q_b(s, \tilde{K}_2(s)) ds)} \left[\mathcal{L}^0(j, t, x(t), [u(j, t), u^*(i, t)]) \right. \\
 &\quad \left. + q_b(t, \tilde{K}_2(t)) V^1(j, t, x(t)) : \text{ given } u^*(i, \cdot); i \neq j \right] dt \\
 &\text{s.t.} \\
 \dot{x}(t) &= f^0(t, x(t), [u(j, t), u^*(i, t)]); \quad u(t) \in U(t); \quad t \geq 0; \quad x(0) = x^0.
 \end{aligned}$$

A similar method is used to compute the Nash-equilibrium of this equivalent deterministic differential game

Initial time control



Outline

- 1 Stochastic Control Approach
 - Economic modeling
 - Climate change dynamics
 - Stochastic control formulation
 - Dynamic programming
 - Numerical results
- 2 Stochastic Game Approach
 - Multi-region modeling
 - Game in normal form
 - Dynamic programming
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- 3 Conclusion

Conclusion

- ① We propose a **stochastic control model** of the world economy for the design of efficient climate policies, under uncertainty related to the availability date of advanced clean technologies.
- ② We propose also a **stochastic game model** to address the problem of non-cooperative behavior of groups of countries in the design of emission abatement strategies, under similar technological uncertainty.